Fourier Transform

* Represent (transform) a non-Periodic Function from time domain to Frequency domain

$$T.D \xrightarrow{F.T} F.D$$

$$g(t) \xrightarrow{F.T} G(F)$$

$$G(F) = \int_{-\infty}^{\infty} g(t) \cdot e^{-j2\pi F t} \cdot dt$$

I.F.T.

$$F.D \xrightarrow{I.F.T.} T.D$$

$$G(F) \xrightarrow{I.F.T} g(t)$$

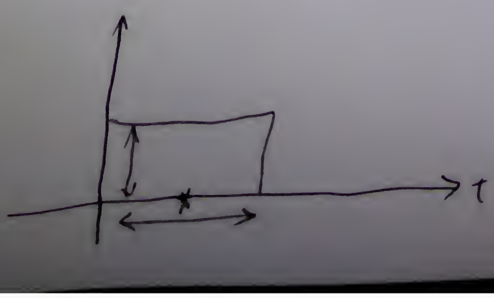
$$g(t) = \int_{-\infty}^{\infty} G(F) \cdot e^{+j2\pi F t} \cdot dF$$

[1] Find F.T of the rectangular pulse shown.

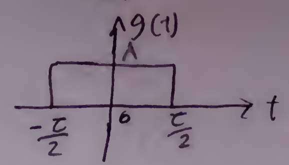
$$g(t) = A\,\Pi\left(\frac{t}{T}\right) = A \text{ rect}\left(\frac{t}{T}\right)$$

to draw rect

1- amplitude (A)
2- Center
تأكد أن البسط مضروب في واحد وساويه بالصفر
3- Width المقام



$$A \text{ rect}\left(\frac{t}{\tau}\right)$$



$$G(f) = \int_{-\infty}^{\infty} g(t) \cdot e^{-j2\pi F t} \, dt$$

$$= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A \cdot e^{-j2\pi f t} \, dt$$

$$= \frac{A}{-j2\pi f}\left[e^{-j2\pi f t}\right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}}$$

$$= \frac{A}{-j2\pi f}\left[e^{-j2\pi f\frac{\tau}{2}} - e^{+j2\pi F\frac{\tau}{2}}\right]$$

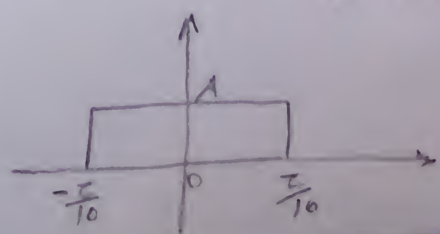
$$= \frac{A}{\pi f}\left[\frac{e^{+j\pi f\tau} - e^{-j\pi f\tau}}{2j}\right]$$

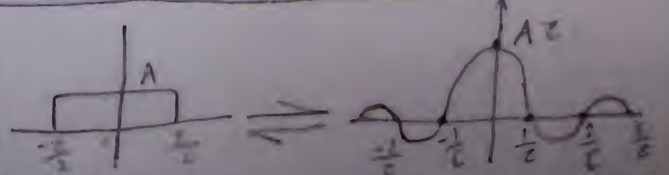
$$= \frac{A\tau}{\pi f\tau}\sin(\pi F\tau)$$

$$= A\tau \cdot \text{sinc}(F \cdot \tau)$$

$$A \text{ rect}\left(\frac{t}{\tau}\right) \rightleftharpoons A\tau \,\text{sinc}[F\tau]$$

$$A \text{ rect}\left(\frac{5t}{\tau}\right) = A \text{ rect}\left[\frac{t}{\tau/5}\right]$$



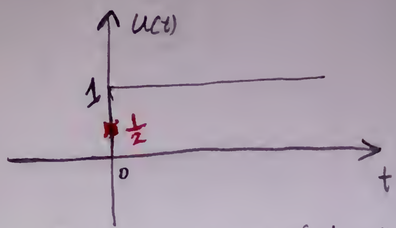


$A \text{ rect}\left(\frac{t}{\tau}\right)$ $\qquad$ $A\tau \text{ sinc}(F\tau)$

* Some Important Fuctions:-

1 Unit Step Function $u(t)$

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

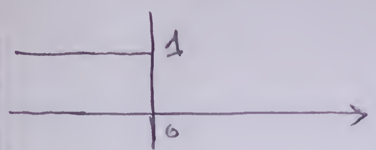


$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases}$$

الصفر له قيمتين لذلك (0,1)
تكون قيمته هي المتوسط

$-u(t)$

$u(-t)$

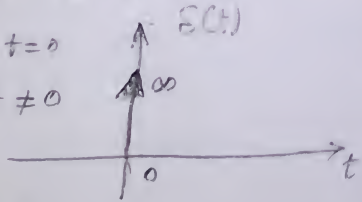


2 Signum Function $sgn(t)$

$$sgn(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

3 Delta Function $\delta(t)$

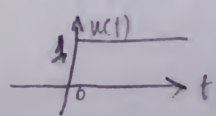
$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

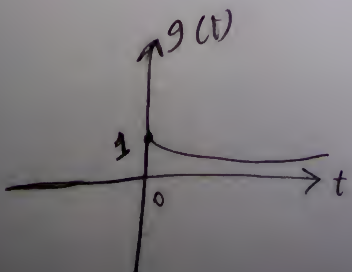


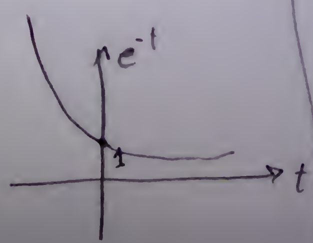
$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

2 Find F.T. For $g(t) = u(t) . e^{-t}$

* Draw







$$G(f) = \int_{-\infty}^{\infty} g(t) . e^{-j2\pi f t}\, dt$$

$$= \int_{0}^{\infty} e^{-t} . e^{-j2\pi f t}\, dt$$

$$= \frac{1}{-j2\pi f - 1} \left[e^{-t(+j2\pi f + 1)} \right]_0^{\infty}$$

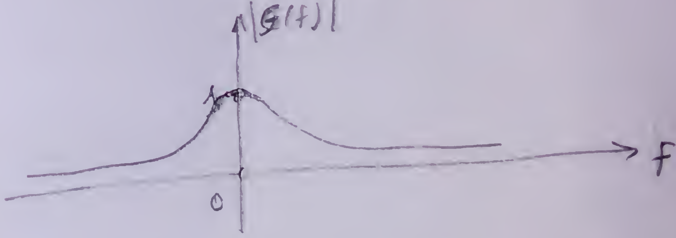
$$= \frac{1}{-(1 + j2\pi f)} \left[e^{-\infty} - e^{0} \right]$$

$$= \frac{1}{1 + j2\pi f}$$

$$u(t) . e^{-t} \rightleftharpoons \frac{1}{1 + j2\pi f}$$

* Draw in F.D

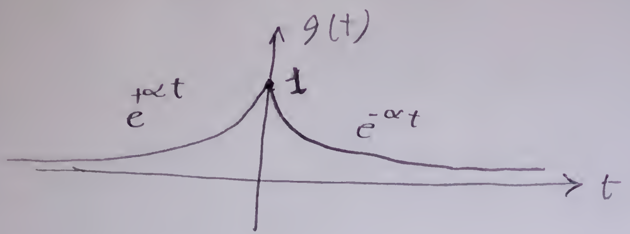
$$|G(f)| = \frac{1}{\sqrt{1 + 4\pi^2 f^2}}$$



***Ex:** Find F.T. of $g(t) = e^{-\alpha|t|}$

$$|t| = \begin{cases} t & \geq 0 \\ -t & < 0 \end{cases}$$

$$g(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \\ e^{\alpha t} & t < 0 \end{cases}$$



$$G(f) = \int_{-\infty}^{\infty} g(t) \cdot e^{-j2\pi ft} \, dt$$

$$= \int_{-\infty}^{0} e^{\alpha t} \cdot e^{-j2\pi ft} \, dt + \int_{0}^{\infty} e^{-\alpha t} \cdot e^{-j2\pi ft} \, dt$$

$$= \left. \frac{e^{t(\alpha - j2\pi f)}}{\alpha - j2\pi f} \right|_{-\infty}^{0} + \left. \frac{e^{-t(\alpha + j2\pi f)}}{-(\alpha + j2\pi f)} \right|_{0}^{\infty}$$

$$= \frac{e^{0} - e^{-\infty}}{\alpha - j2\pi f} - \frac{e^{-\infty} - e^{0}}{\alpha + j2\pi f}$$

$$= \frac{1}{\alpha - j2\pi f} + \frac{1}{\alpha + j2\pi f}$$

$$= \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}$$

$$e^{-\alpha|t|} \rightleftharpoons \frac{2\alpha}{\alpha^2 + 4\pi^2 F^2}$$

* Properties of Fourier Transform

هي مجموعة من الخواص تساعدنا في إيجاد F.T. لدوال مجهولة بمعلومية دوال أخرى وليس عن طريق التكامل

1 Linearity (Super Position)

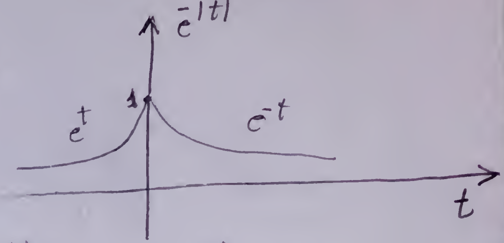
Let $g_1(t) \rightleftharpoons G_1(f)$

$g_2(t) \rightleftharpoons G_2(f)$

$$a\, g_1(t) \pm b\, g_2(t) \rightleftharpoons a\, G_1(f) \pm b\, G_2(f)$$

Ex: Find F.T. for $g(t) = e^{-|t|}$

Using Linearity



$$g(t) = e^{+t} u(-t) + e^{-t} u(t)$$

$$\because e^{-t} \cdot u(t) \rightleftharpoons \frac{1}{1 + j2\pi f}$$

$$\because e^{t} \cdot u(-t) \rightleftharpoons \frac{1}{1 - j2\pi f}$$

$$\therefore G(f) = \frac{1}{1 + j2\pi f} + \frac{1}{1 - j2\pi F}$$

2 Time Scaling

$$g(t) \rightleftharpoons G(f)$$

$$g(at) \rightleftharpoons \frac{1}{|a|} G\left(\frac{f}{a}\right)$$

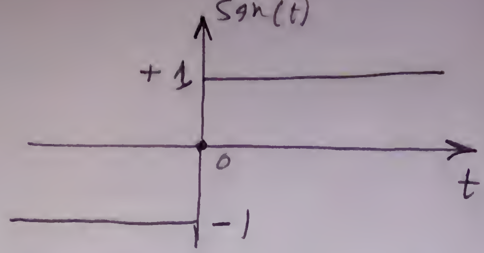
Ex: $g(t) = e^{-at} \cdot u(t)$ $\quad u(at) = u(t)$

$$\because e^{-t} \cdot u(t) \rightleftharpoons \frac{1}{1 + j2\pi f}$$

Using Time Scaling

$$\therefore g(t) = e^{-at} u(at) \rightleftharpoons \frac{1}{|a|} \cdot \frac{1}{1 + j2\pi \frac{f}{a}}$$

Ex: $g(t) = sgn(t)$



$$G(f) = \int_{-\infty}^{0} -1 \cdot e^{-j2\pi ft}\, dt + \int_{0}^{\infty} 1 \cdot e^{-j2\pi ft}\, dt$$

$$= -\frac{e^{-j2\pi ft}}{-j2\pi f}\Big|_{-\infty}^{0} + \frac{e^{-j2\pi ft}}{-j2\pi f}\Big|_{0}^{\infty}$$

$$= \frac{[e^{0} - e^{\infty}]}{j2\pi f} + \frac{e^{-\infty} - e^{0}}{-j2\pi f}$$

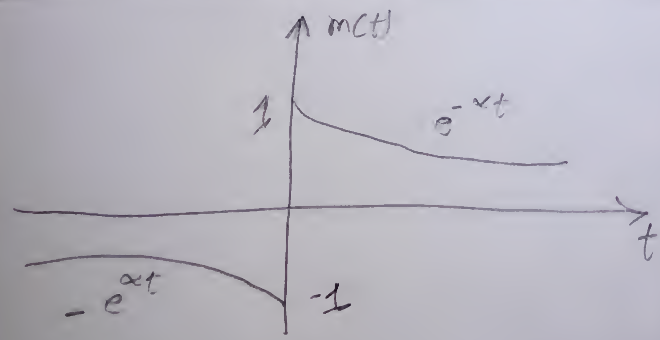
$$= \infty$$

لايمكن حله بالطريقة العادية

$m(t)$

$$sgn(t) = \lim_{\alpha \to 0} [m(t)]$$

$$F.T[sgn(t)] = \lim_{\alpha \to 0} [M(f)]$$



$$sgn(t) = \lim_{\alpha \to 0} m(t)$$

$$m(t) = -e^{\alpha t} \cdot u(-t) + e^{-\alpha t} u(t)$$

$$\because e^{-\alpha t} \cdot u(t) \rightleftharpoons \frac{1}{\alpha + j2\pi f}$$

$$\because -e^{\alpha t} \cdot u(-t) \rightleftharpoons -\frac{1}{\alpha - j2\pi f}$$

using super position

$$M(f) = \frac{1}{\alpha + j2\pi f} - \frac{1}{\alpha - j2\pi f}$$

$$= \frac{-j4\pi f}{\alpha^2 + 4\pi^2 f^2}$$

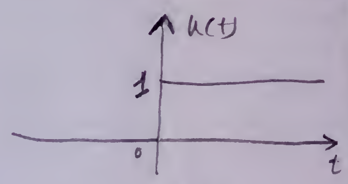
$$F.T.[sgn(t)] = \lim_{\alpha \to 0} \frac{-j4\pi f}{\alpha^2 + 4\pi^2 f^2}$$

$$G(f) = -j \frac{1}{\pi f}$$

$$|G(f)| = \frac{1}{\pi f}$$

$$sgn(t) \rightleftharpoons \frac{1}{j\pi f}$$

Ex: $g(t) = u(t)$

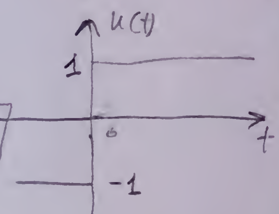


$$sgn \rightleftharpoons \frac{1}{j\pi f}$$

$$u(t) = \frac{1}{2}[sgn(t) + 1]$$

Using super position

$$u(t) \rightleftharpoons \frac{1}{2}\left[\frac{1}{j\pi f} + \frac{1}{2}\delta(f)\right]$$



$$\boxed{a \rightleftharpoons a\,\delta(f)}$$